NUMERICAL TREATMENT OF PRESTRESSING TENDONS IN THE NONLINEAR ANALYSIS OF PRESTRESSED CONCRETE STRUCTURES

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Abstract—A formulation for the nonlinear material and geometric, instantaneous and long-term analysis of prestressed concrete structures is described, where the treatment of prestressing is achieved as a further extension of existing numerical models for reinforced concrete structures. Both the geometric and mechanical aspects of prestressing are treated in a consistent way with respect to the displacement formulation of the finite element method which is usually adopted for such analyses. A brief review of some common treatments for many of the existing numerical models for reinforced concrete beams, slabs and shells is made as well as the description of both the theoretical and computational features that are proposed for their extension to include prestressing. Special focus is given to such aspects as the geometric description of the layout of the tendons, force losses, prestress equivalent forces, prestress strain increments under external loads, the contribution of the prestressing to the global stiffness, and the stress relaxation of steel.

1. INTRODUCTION AND SCOPE

During the last decade, a considerable amount of research effort was devoted to the development of numerical models for the nonlinear behaviour of reinforced concrete structures. Special attention was paid to the application of such models to particular structural types, such as beams, slabs and shells. Most of these numerical models were conceived by coupling a displacement formulation of the finite element model together with a set of partial constitutive models for the main aspects of the nonlinear behaviour of the materials, such as the actual stress–strain relationships, the initiation and further propagation of cracks in concrete, bond between reinforcing steel and concrete, and the time-dependent phenomena such as creep and shrinkage of concrete, among others. In addition, these models included the nonlinear geometric effects caused by large displacements by introducing the equilibrium condition in the deformed geometry of the structure and by also considering the displacement quadratic terms of the compatibility equations.

These constructions, once implemented in the computer and used systematically, could be adequately verified by comparison with the available experimental results. This was achieved by Kang [1], Van Greunen [2], Floegel [3], Chan [4] and Mari [5] in their respective works, among other researchers.

Recently, the popularization of sophisticated constructive techniques has promoted the extension of the existing models to new capabilities. The inclusion of the prestressing effects might be a significant example of such further developments. Most of the formulations presented up to now which include prestressing have been developed as an extension of existing models for the analysis of reinforced concrete structures, and agree in their main features [7–9]. However, so far this application has found a practical limitation in the lack of available experimental results to be used for the verification.

This paper describes a formulation to include the prestressing effects in available numerical models for the nonlinear geometric and material, instantaneous and long-term analysis of reinforced concrete beams, slabs and shells, based on the displacement formulation of the finite element method. The geometric and mechanic aspects of prestressing are treated in a unified way with regard to the following aims: (1) the search of a maximum level of automatism and, (2) consistency with the finite element method technique.

A complete description, including detailed mathematic derivations, can be found in [8]. In a companion paper, the application of this formulation to the analysis of prestressed concrete shells of general geometry is discussed together with its verification by comparison with experimental studies from other authors [10]. A first application to prestressed concrete axisymmetric shells has already been presented [11].

2. FINITE ELEMENT FORMULATION FOR THE NONLINEAR ANALYSIS OF BEAMS, SLABS AND SHELLS

The following sections contain a brief review of the common features of the general formulation of most of the nonlinear geometric and material analyses of
Fig. 1. Displacement of a point in a shell element due to the rotation of the orthogonal line to the middle surface.

beams and shell structures of concrete. A slab is regarded as a particular case of a shell structure in this paper.1 This review is intended to introduce the concepts, notation and criteria which will be needed to describe the proposed method for the inclusion of prestressing.

2.1. Interpolation of the displacement field of a shell element

The finite element formulations specifically developed to treat shell structures, such as Ahmad's [13] and other similar ones subsequently presented,‡ obtain an important advantage in both efficiency and accuracy by assuming that the straight lines initially normal to the shell middle surface remain straight after the deformation of the structure. This assumption allows the definition of the interpolation for the displacement field with variables, displacements and rotations which are wholly related to the middle surface. Consequently, a shell finite element can be chosen which includes all its nodes in the middle surface as well.

This assumption also establishes a direct relationship between the displacement vector \( \mathbf{u} \) at any point of the shell, the displacement \( u_0 \) of its orthogonal projection to the middle surface, and the rotations of the normal \( \alpha_1 \) and \( \alpha_2 \), defined according to a set of surface orthogonal vectors (Fig. 1)

\[
\mathbf{u} = u_0 + \xi \alpha_2 w_1 - \xi \alpha_2 v_2.
\] (1)

The components of the displacement vector at the middle surface, \( u_0 = (u_{0x}, v_{0y}, w_{0z}) \), together with the rotations \( (\alpha_{1x}, \alpha_{2z}) \) set up five degrees of freedom for each node of the element.

It is useful to express the displacement field interpolation through a set of shape functions \( N_i \), \( i = 1, \ldots, N \) formulated with two natural coordinates in the plane of the shell \((\xi, \eta)\). For an isoparametric element, the same shape functions will be used to interpolate the geometry from the nodal coordinates. Even a variable thickness may be described by means of such functions by providing the thickness at each node \( \{t_i, i = 1, 2, \ldots, N\} \). The position \((x)\) of an arbitrary point of the shell is thus obtained as

\[
x = \sum_{i=1}^{N} (\xi, \eta) x_{0i} + \zeta \sum_{i=1}^{N} N_i(\xi, \eta) t_i y_{0i},
\] (2)

where \( \zeta (|\zeta| \leq 1) \) is a third natural coordinate which characterizes the distance to the middle surface.

By introducing the shape functions in (2) the following expression is obtained

\[
\mathbf{u} = \sum_{i=1}^{N} N_i(\xi, \eta) u_{0i} + \zeta \sum_{i=1}^{N} N_i(\xi, \eta) t_i [-v_2, v_1, \alpha_{1i} \alpha_{2i}]^T
\] (3)

or, in a compact notation

\[
\mathbf{u} = \Theta \mathbf{a},
\] (4)

where \( \mathbf{a} \) is defined as a vector which includes all the movement components (displacements and rotations) of all the nodes of the element

\[
\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}, \quad \mathbf{a}_i = \begin{bmatrix} u_{0i} \\ v_{0i} \\ w_{0i} \\ \alpha_{1i} \\ \alpha_{2i} \end{bmatrix}, \quad i = 1, 2, \ldots, N
\] (5)

and \( \Theta \) is a matrix obtained as

\[
\Theta = [\Theta_1, \Theta_2, \ldots, \Theta_N],
\] (6)

where each submatrix \( \Theta_i, i = 1, 2, \ldots, N \) characterizes the contribution of each node to the interpolation of the displacement field. Matrix \( \Theta_i \) has the following explicit form

\[
\Theta_i = \begin{bmatrix}
N_i & 0 & 0 & -\frac{\zeta}{2} N_i t_i (v_{2i})_x & \frac{\zeta}{2} N_i t_i (v_{1i})_x \\
0 & N_i & 0 & -\frac{\zeta}{2} N_i t_i (v_{1i})_y & \frac{\zeta}{2} N_i t_i (v_{2i})_y \\
0 & 0 & N_i & -\frac{\zeta}{2} N_i t_i (v_{1i})_z & \frac{\zeta}{2} N_i t_i (v_{2i})_z 
\end{bmatrix}
\] (7)

Alternative formulations to Ahmad's [13] may yield explicit expressions for the displacement interpolation different to (7). However, its compact form will fit in general with eqn (4).

As is well known, the equilibrium condition, together with the kinematic compatibility and the
constitutive relationships, completely determines a structural problem. Yet the finite element method adds a new constraint by restricting the movement of the structure to a predefined interpolation of the nodal variables. From a restricted point of view, this condition is excessive and must produce a sort of incompatibility with the remaining conditions. In effect, when introducing the equilibrium condition by the virtual work principle, a set of equations results where coefficients are obtained through integration of the whole volume of each element. As a consequence, the equilibrium is only guaranteed at an interelemental level.

2.3. Strain distribution

The strain state of a certain point can be determined by a vector $\epsilon$ which includes the six different components $\epsilon_{ij}$ of the strain tensor

$$\epsilon = \{\epsilon_{x}, \epsilon_{y}, \epsilon_{x y}, \epsilon_{x z}, \epsilon_{y z}, \epsilon_{z z}\}^T.$$  

The strain components are obtained from the movements through the conditions of kinematic compatibility, including their quadratic terms to account for finite displacements

$$\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \sum_{k=1}^{3} \left( \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right).$$

It is convenient, for later purposes, to distinguish between the linear $\epsilon$ and quadratic $\eta$ parts so that

$$\epsilon = \epsilon + \eta.$$  

Using these equations, it is possible to build an operator $L$ which allows the linear part to be obtained as

$$\epsilon = Lu.$$  

Using expression (4) for the displacement field, it follows that

$$\epsilon = Ba, \quad B = L\Theta$$

thus resulting in a direct determination of the strains from the nodal variables.

2.4. Stress–strain relationship

Vector $\sigma$ is defined as the one which includes the six different components of the stress tensor

$$\sigma = \{\sigma_{x}, \sigma_{y}, \tau_{xy}, \tau_{xz}, \tau_{yz}, \tau_{zz}\}^T.$$  

If nonlinear constitutive models are used to reproduce the behaviour of the materials, the stress–strain relationship will acquire complex mathematical expressions or possibly will not produce explicit formulations. Some models establish such a relationship through a secant matrix. When accounting for initial stresses ($\sigma_0$) and initial strains ($\epsilon_0$), a model of this type will yield a matrix expression such as

$$\sigma = D(\epsilon - \epsilon_0) + \sigma_0.$$
For an incremental model, eqn (16) is still valid if the stresses and strains are actually regarded as increments while \( D \) is regarded as a tangent matrix of mechanical properties.

Again, the preponderance of some dimensions in beams, slabs and shells allows the introduction of additional hypotheses. Formulations for slabs and shells usually assume that the stress component normal to the middle surface \( (\sigma_n) \) is almost null and can be neglected. As a consequence, the corresponding strain \( (\varepsilon_n) \) does not participate in the strain energy of the system and there is no need to include it in the formulation.

2.5. Virtual work equation

The equilibrium condition is introduced by means of the virtual work equation. A relationship results between the external forces and the internal stress distribution.

The system of external forces will be assumed to include volumetric loads \( r \), surface loads \( r_s \), and a set of concentrated loads \( R_j \), \( i = 1, 2, \ldots, n \), applied at certain points \( x_i \), \( i = 1, 2, \ldots, n \) of the element.

The virtual work produced by the external forces is

\[
\delta W_e = \int_V \delta \mathbf{u}^T \mathbf{r} \, dV + \int_A \delta \mathbf{u}^T \mathbf{r}_s \, dA + \sum_{i=1}^N \delta u(x_i)^T \mathbf{R}_i
\]

while virtual work related to the internal state of stress is

\[
\delta W_i = \int_V \delta \mathbf{e}^T \sigma \, dV = \int_V \varepsilon^T \sigma \, dV + \int_V \eta \, dV \mathbf{e}^T \sigma
\]

where \( \delta \mathbf{e} \) are the virtual strains which correspond to the virtual displacement \( \delta \mathbf{u} \).

By applying the virtual work equation to an individual finite element together with eqns (4), (14) and (16), the following condition is obtained

\[
\delta \mathbf{a}^T \left[ \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} \, dV \right] + \delta \mathbf{a}^T \int_V \mathbf{B}^T \mathbf{\sigma}_0 \, dV - \delta \mathbf{a}^T \times \int_V \mathbf{B}^T \mathbf{D} \mathbf{e}_0 \, dV = \delta \mathbf{a}^T \int_A \mathbf{\Theta}^T \mathbf{r} \, dV
\]

\[
+ \delta \mathbf{a}^T \int_A \mathbf{\Theta}^T \mathbf{r}_s \, dA + \delta \mathbf{a}^T \sum_{i=1}^N \mathbf{\Theta}(x_i)^T \mathbf{R}_i
\]

where \( \delta \mathbf{a} \) is a vector of virtual nodal movements.

As described by Zienkiewicz [12], the term due to the quadratic part of this strains can be transformed in the following way

\[
\int_V \delta \mathbf{e}^T \mathbf{r} \, dV = \delta \mathbf{a}^T \int_V \mathbf{G}^T \mathbf{S} \, dV \mathbf{a},
\]

where \( \mathbf{G}^T \) is a linear differential operator and \( \mathbf{S} \) is a square symmetric matrix that contains components of the stress vector \( \mathbf{r} \).

The displacement fields obtained from (4) verify systematically the kinematic compatibility. As a consequence, eqn (19) must be true for all arbitrary values of \( \delta \mathbf{a} \), thus requiring the following condition

\[
\left[ \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} \, dV + \int_V \mathbf{G}^T \mathbf{S} \, dV \right] \mathbf{a} + \int_V \mathbf{B}^T \mathbf{\sigma}_0 \, dV
\]

\[
- \int_V \mathbf{B}^T \mathbf{D} \mathbf{e}_0 \, dV = \int_V \mathbf{\Theta}^T \mathbf{r} \, dV + \int_A \mathbf{\Theta}^T \mathbf{r}_s \, dA + \sum_{i=1}^N \mathbf{\Theta}(x_i)^T \mathbf{R}_i.
\]

This condition can be expressed in a more compact notation as

\[
(K_0 + K_o) \mathbf{a} + \mathbf{R}_i + \mathbf{R}_j = \mathbf{R},
\]

where \( K_0 \) and \( K_o \) are the material and the geometric stiffness matrices respectively. The terms \( \mathbf{R}_i \) and \( \mathbf{R}_j \) can be regarded as fictitious load vectors related to initial stresses and the initial strains.

Finally

\[
\mathbf{R} = \int_V \mathbf{\Theta}^T \mathbf{r} \, dV + \int_A \mathbf{\Theta}^T \mathbf{r}_s \, dA + \sum_{i=1}^N \mathbf{\Theta}(x_i)^T \mathbf{R}_i
\]

is a vector of external equivalent nodal loads. Equation (23) constitutes a process by which the actual external loads are transformed into an equivalent system of concentrated forces applied at the nodes.

2.6. Geometric nonlinearity

Nonlinear geometric effects are caused by the consideration of finite movements, both assuming large displacements and finite rotations. First, quadratic terms of the compatibility equations are included and treated as shown in Secs 2.3 and 2.5. Furthermore, the geometry of the structure is continuously actualized by adding the displacement increments to the current coordinates of the nodes, according to the updated Lagrangian description. This asks for an iterative procedure until convergence is obtained meaning that the equilibrium condition has been finally imposed on the deformed geometry of the structure.

Some analyses of three-dimensional shells or beams assume small incremental rotations to diminish the dependence in the order of the orthogonal transformations needed to update the orientation of the local reference axes.

2.7. Reinforced concrete elements for nonlinear material analysis

Generally, the volume integrations repeatedly asked by the finite element technique according to
Sec. 2.5, are to be solved by means of adequate numerical rules. It is useful to distinguish between surface and thickness integrations for shells, or cross-sectional and axial integrations for beams, thus allowing the selection of more appropriate numerical rules.

Both the variation of the mechanical properties of the materials throughout the loading process, as well as the modelling of the steel reinforcement, are accounted for through the integration on the volume of the elements. Each integration point in the thickness of a shell (or in the cross-section of a beam) is regarded with a specific material state associated with a certain set of mechanical properties. Additional integration points are defined to account for reinforcement.

2.8. Time-dependent effects

The nonmechanical strains ($\varepsilon^{nm}$) are introduced and defined as the part of the total strains ($\varepsilon$), which are obtained in addition to that which would have obtained in an instantaneous analysis ($\varepsilon^m$)

$$\varepsilon = \varepsilon^m + \varepsilon^{nm}. \quad (24)$$

Formally, the nonmechanical strains ($\varepsilon^{nm}$) can be treated as the initial strains ($\varepsilon_0$) of eqn (21), so that the equilibrium condition, as introduced by the virtual work equation, produces a term of fictitious forces

$$R_0^{nm} = -\int_{V_0} B^T D^e \Delta \varepsilon^{nm} dV. \quad (25)$$

The following individual contributions are considered as part of the nonmechanical strains: concrete creep $\Delta \varepsilon^c$, concrete shrinkage $\Delta \varepsilon^s$, concrete ageing $\Delta \varepsilon^a$, and possible strains due to thermal effects $\Delta \varepsilon^t$. Just as an approach, superposition is assumed thus neglecting their actual coupling

$$\Delta \varepsilon^{nm} = \Delta \varepsilon^c + \Delta \varepsilon^s + \Delta \varepsilon^a + \Delta \varepsilon^t. \quad (26)$$

2.9. Solution strategy for nonlinear analysis

The nonlinear material and geometric problem needs a solution strategy to carry out the analysis throughout all the loading process. Usually, the strategy consists of the solution of a series of individual linear problems such as (22), which are generated in a recursive way by introducing the resulting actual state of stress $\sigma$ as the new initial stress $\sigma_0 + \varepsilon$ in eqn (21) for the subsequent iteration. Thus, the term associated to the stresses in (21) is actually a vector of internal resisting forces.

The difference between the applied total external loads $R$ and the internal resisting forces $R^i$ are the unbalanced or residual forces $R^u$ which remain still unapplied at a certain step of the resolution process. Convergence implies the dissipation of these residual forces.

3. PROPOSED METHODOLOGY FOR THE INCLUSION OF PRESTRESSING

3.1. Geometric description of the layout of a tendon

The following description, first proposed by Hofstetter [7], has been adopted in this formulation for a shell prestressing tendon

$$x_p = x + e(s)v_1, \quad (27)$$

where $x_p$ refers to the position of the axis of the tendon, $x_0$ refers to the middle surface of the shell, and $v_1$ is the unit vector orthogonal to that. Function $e(s)$ is used to describe the eccentricity of the tendon with respect to the middle surface of the shell (Fig. 3).

Similarly, to describe a tendon included in a three-dimensional curved beam, two parametric functions $e_x(s)$ and $e_z(s)$ may be used to build the following representation

$$x_p = x_0 + e_x(s)v_2 + e_z(s)v_3, \quad (28)$$

where $x_0$ refers to the centroidal axis of the beam and $v_2, v_3$ are two unit vectors associated to the local reference axes $y'$ and $z'$ in the plane of the cross-section (Fig. 2).

3.2. Equilibrium of a space cable

Figure 4 represents a differential segment of a three-dimensional cable, submitted to a set of linear distributed forces $p_t$ tangent to the axis of the cable, $p_n$ perpendicular to it and permanently directed towards the local centre of curvature, and $p_b$, parallel

![Fig. 4. Equilibrium of a space cable segment.](image)
to the binormal unit vector. Due to the null bending stiffness, only an axial force \( P(s) \) can exist at any point of the cable. The equilibrium condition can be written as

\[
\int_{s_1}^{s_2} \left( P(s) - \frac{1}{2} \frac{dP}{ds} \right) ds + \left( P(s) + \frac{1}{2} \frac{dP}{ds} \right) ds + (p_t + p_n + p_b) ds = 0. \tag{29}
\]

Operating and neglecting the second order terms yields

\[
P \frac{dt}{ds} + \frac{dP}{ds} t = -p_t - p_n - p_b. \tag{30}\]

Using the following geometric relationship

\[
\frac{dt}{ds} = \kappa n, \tag{31}\]

where \( \kappa \) is the local curvature, gives

\[
P \kappa n + \frac{dP}{ds} t = -p_t - p_n - p_b \tag{32}\]

or, in a scalar form

\[
p_n = -\kappa P \tag{33}\]

\[
p_t = -\frac{dP}{ds} \tag{34}\]

\[
p_b = 0. \tag{35}\]

### 3.3. Prestressing force and losses

As a previous step to the aim of obtaining the work-equivalent forces from prestress, the actual distribution of prestressing force throughout the tendon must be computed taking into account the instantaneous force losses due to friction and anchor slip. Firstly, some physical aspects are introduced and secondly, numerical procedures developed to evaluate the losses are discussed.

#### 3.3.1. Friction between the tendon and the duct

A kinematic friction between the strands and the duct will produce friction interaction forces \( p_f \) between the concrete and the tendon. The most common approach assumes those forces to be simply proportional to the pressure \( p_n \)

\[
p_f = \pm \mu p_n, \tag{36}\]

where the sign of the sliding friction coefficient \( \mu \) depends in its turn upon the sign of the sliding motion of the tendon relative to the surrounding concrete. A positive sign must be used when the direction of the total accumulated length of the tendon coincides with the direction of the sliding motion.

The coupling of eqns (33), (34) and (36) yields the differential equation

\[
\frac{dP}{ds} = \pm \mu \kappa P \tag{37}\]

which solution leads to the well-known formula to relate prestressing forces between any two points \( s_1 \) and \( s_2 \)

\[
P(s_2) = P(s_1) \exp \left( \pm \mu \int_{s_1}^{s_2} \kappa ds \right). \tag{38}\]

Wobble of the tendon in the duct is simply modelled by adding an additional empiric value \( K_w \) to the geometric curvature \( \kappa \). The term \( K = \mu K_w \) known as the wobble friction coefficient.

#### 3.3.2. Loss due to pull-in of wedges

At the end of the jacking process, after having extended the tendon to the specified load and elongation, the jack is used to advance the anchorage wedges against the anchor plate until seating them forcibly. This process produces a certain recovery on the initial length of the tendon with a subsequent loss of force. The draw-in of the wedges depends upon the anchorage system and the diameter of the strands.

This movement produces a reverse friction effect that frequently constrains the loss to a limited zone close to the anchorage. Let \( Ad \) be the total value of pull-in of wedges and \( l_s \) the length over which the

![Fig. 5. Prestressing force distribution throughout the length of a tendon.](image)

![Fig. 6. Prestressing force distribution when wedge pull-in losses reach the opposite end.](image)
corresponding losses extend. The condition for the determination of the influenced length \( l_a \) is

\[
\Delta d_a = \int_0^{l_a} \Delta \epsilon_a(s) \, ds = \frac{1}{E_p A_p} \int_0^{l_a} \Delta P(s) \, ds, \tag{39}
\]

where \( \Delta \epsilon_a(s) \) is the tendon incremental strain due to the wedge pull-in, \( A_p \) is the cross-sectional area of prestressing steel, and \( E_p \) is the elastic modulus of the prestressing steel.

As shown by Fig. 5, eqn (39) can also be written as

\[
\Delta d_a = \frac{1}{E_p A_p} \int_0^{l_a} (P'(s) - P_3(s)) \, ds. \tag{40}
\]

Using eqn (38) repeatedly until an explicit expression is obtained for (40), the following transcendental equation for the unknown \( l_a \) results

\[
\Delta d_a = \int_0^{l_a} \left[ \exp\left(-\mu \int_0^s \kappa^a \, ds\right) - \exp\left(-2\mu \int_0^s \kappa^a \, ds\right) \exp\left(\mu \int_0^s \kappa^a \, ds\right) \right] \, ds. \tag{41}
\]

Eventually, a large pull-in may produce a shortening effect which reaches the opposite end. In such a situation the integration (39) must be extended over the whole length of the tendon and coincides with the area enclosed by \( AA' \) and \( CC' \) curves in Fig. 6. The only unknown, \( P_a \), can easily be determined from

\[
\Delta d_a = \frac{1}{E_p A_p} \int_0^{l_a} \left[ \exp\left(-\mu \int_0^s \kappa^a \, ds\right) - \exp\left(\mu \int_0^s \kappa^a \, ds\right) \exp\left(-\mu \int_0^s \kappa^a \, ds\right) \right] \, ds. \tag{42}
\]

3.4. Numerical evaluation of losses due to friction and wedge pull-in

The geometric and mechanical properties of a tendon which are needed throughout the analysis, are evaluated at a discrete number of predefined integration points \( s_{k_i}, i = 1, 2, \ldots, n \). Also the prestressing force must be evaluated only at those points using eqn (38) recursively.

The calculation of the length \( l_a \) over which the losses due to edge pull-in extend, is performed by a direct method, independently developed by Mari [5] and Hofstetter [7]. Value of \( \Delta d^a \) function, as defined below, is evaluated at the successive points \( s_{k_i}, i = 1, 2, \ldots, n \) starting from the anchorage being jacked. At each step, \( l_a \) is made equal to \( s_{k_i} \)

\[
\Delta d^a = \int_{s_{k_{i-1}}}^{s_{k_i}} P'(s) - P_3(s) \, ds. \tag{43}
\]

The condition

\[
\phi_i - (\Delta d^a_{i-1} - \Delta d) (\Delta d^a - \Delta d) < 0 \tag{44}
\]

means that the actual length \( l_a \) is found between the consecutive points \( s_{k_{i-1}} \) and \( s_{k_i} \). It can then be approached by a linear interpolation between these two boundaries.

Condition (44) may not be accomplished at any point throughout the tendon, meaning that the losses extend until the opposite end. Equation (42) must be used in this case to determine the force drop at end, \( P_C \).

The final prestressing force distribution is obtained as the upper evolvent of the curves independently computed for friction and edge pull-in from each jacking end \( CC'/ED'D \) in Fig. 5. This requires a further process of comparison through all the integration points.

3.5. Equivalent node forces caused by prestress

Once the prestressing force \( P(s) \) is known, vector \( p \) can be constructed to describe the linear distributed forces produced by the prestressing tendon on the remaining part of the structure, as

\[
p(s) = \frac{dP(s)}{ds} \, t + \kappa^p P(s) \, n. \tag{45}
\]

The process shown by eqn (24) is used to obtain the consistent node element forces \( P \) (Fig. 7) related to the distributed linear load \( p \). This yields a curvilinear integration along the tendon axial curve segment \( \Gamma \) included into an element

\[
P = \int_{\Gamma} \Theta^T p(s) \, ds. \tag{46}
\]

Doubts may arise regarding the capability of the process to produce a global system of prestress node forces which really verifies the expected self-balanced condition. Neither the geometry of the finite element model nor the integration procedures to be used are exact. Particularly, a shell geometry described by means of independent element numeric interpolations like (2) may not have unique slopes at the boundaries between elements (Fig. 8).

However, global self-balanced conditions can be guaranteed by forcing its accomplishment at each individual prestressing segment included in an
Fig. 8. Prestressing self-correcting concentrated forces at the element interfaces.

element. Two complementary procedures are used bearing the following in mind:

1. Prestressing produces a set of interacting forces between the concrete elements which may be synthesized into two concentrated forces at the ends of the tendon segment, \( P_{M+1} \) and \( P_N \) (Fig. 7). By adding these forces to the prestressing node forces \( P \), an elemental self-balanced nodal force system should be obtained. When adding all the contributions of contiguous elements to form the global system, end segment forces \( PM, i+1 \) and \( PN, i \) should cancel reciprocally. In spite of this, the geometric imperfection at the interface produces a force \( PR \) which has the effect of self-correcting the evaluated set of forces. It follows the convenience of systematically considering the end segment forces thus computing the prestress node forces as

\[
P = \int_R \Theta \left( \frac{dP(s)}{ds} \right) \mathbf{t} + \kappa P(s) \mathbf{n} \ ds
+ P_M \Theta \mathbf{t}_M \mathbf{t}_M^t - P_N \Theta \mathbf{t}_N \mathbf{t}_N^t. \tag{47}
\]

Tendon anchor forces will be directly accounted for in this process.

2. To account for errors due to the inaccuracy of the numerical integration, a correction process has been developed which, based on equilibrium considerations, slightly modifies the individual nodal components to reinforce the self-balanced condition. Its complete description can be found in [8]. In fact, its repeated usage has shown such errors to be significant for most practical cases when the expression (47) is evaluated numerically using a three-point Gauss-Legendre integration scheme.

3.6. Treatment of pre-tensioned and post-tensioned tendons

Distinction between pre-tensioned and post-tensioned reinforcement is achieved through the control of the bond between concrete and prestressing steel. The modelling of the mechanical effects of bonding require the following steps: (1) the inclusion of the prestressing contribution to the global stiffness, (2) the evaluation of the prestress strain increments which are due to external effects on the structure, and (3) the evaluation of the tendon final forces and the generation of the prestress internal resisting forces.

In fact, step 1 will only speed up the convergence of the process since, as it is well known, most of the resolution procedures do not ask for any strictly exact stiffness matrix.

For a pre-tensioned tendon, the bond is permanently considered so that steps (1)–(3) are performed at any load increment. Particularly, prestress strain increments are evaluated when the prestress node forces are applied to simulate the stressing. As a consequence, losses due to elastic compression of the concrete at stressing are automatically accounted for.

The modelling of a post-tensioned tendon is performed in two stages:

1. During all steps up to prestressing transfer (this included), the bond between concrete and steel is not considered. No strain increment in the prestressing steel is assumed to occur prior to the stressing.

2. For later steps some kind of bond exists and steps (1)–(3) are systematically performed. Distinction between bonded and unbonded reinforcement follows through a different calculation of the stiffnesses and the strains produced by external loads, as will be discussed in Secs 3.7–3.10.

3.7. Strain determination for bonded tendons

The displacement interpolation (4) can be used to describe the axial movement \( u_r(s) \) of a bonded prestressing segment into a shell element

\[
u_r(s) = \Theta \{x_r(s)\} \mathbf{a}. \tag{48}\]

Assuming a perfect bond between the concrete and the prestressing steel, eqn (4) can be used in a similar way to evaluate the axial deformation \( \epsilon_a \) of the tendon. The strain vector \( \epsilon \) can be transformed to a different reference axis by means of matrix \( \mathbf{T} \) according to

\[
\epsilon' = \mathbf{T} \epsilon. \tag{49}\]

Matrix \( \mathbf{T} \) may be obtained from a proper reorganization of the tensorial expression \( \epsilon' = t_{ij} \epsilon_i \epsilon_j \), where \( \{t_{ij}\} \) are the direction cosines which describe the new axes.

Particularly, relation (49) can be used to transform the concrete element strains \( \epsilon \) to the axes of the moving trihedral \( t, n, b \) at a point \( P \) of the axial curve of a tendon (Fig. 9). Noting that only the axial deformation of the tendon \( \epsilon_a \) needs to be taken into account, a submatrix \( \mathbf{C} \) can be selected from (49) so that

\[
\epsilon_a = \mathbf{C} \epsilon. \tag{50}\]

Using (48)

\[
\epsilon_a = \mathbf{C} \mathbf{B} \{x_r(s)\} \mathbf{a}. \tag{51}\]

Fig. 9. Axial deformation of a tendon.
Thus, a matrix $B_p, (B_p = CB)$ can be defined which directly relates the tendon axial deformation with the element nodal variables

$$\varepsilon_p = B_p(s)a. \tag{52}$$

### 3.8. Prestressing contribution to global stiffness for bonded tendons

From the application of the virtual work equation to a prestressing tendon, mathematical expressions are obtained to compute its contribution to the stiffness and the internal resisting forces. The process to be followed is similar to that of Sec. 2.5 and is applied to a prestressing segment included in a shell element.

The virtual axial movement of the tendon can be related to the arbitrary nodal movements of the shell $\delta a$ using eqn (52)

$$\delta\varepsilon_p = B_p\delta a. \tag{53}$$

Assuming that an initial axial stress $\sigma_{\mu 0}$ exists, and developing as in Sec. 2.5, the following equation is obtained

$$[I_A + E_p B_p \otimes B_p ds] + A_p \int \sigma_{\mu 0} B_p^T ds = - \int \Theta^T p'' ds - P_{\mu M} \Theta^T t_M + P_{\mu N} \Theta^T t_N, \tag{59}$$

where $-p''$, $-P_{\mu M} t_M$ and $P_{\mu N} t_N$ are the forces produced by the concrete element on the prestressing segment seen as a free body.

The first term

$$K_p = A_p \int E_p B_p^T \otimes B_p ds \tag{60}$$

is a stiffness matrix for the prestressing segment. Although it includes all the degrees of freedom of a shell element, matrix $K_p$ reproduces only the axial stiffness of a curved cable, and will be singular. Matrix $K_p$ must be rather understood as a contribution to the stiffness of a shell element.

The term

$$R_p = A_p \int \sigma_{\mu 0} B_p^T ds \tag{61}$$

corresponds to the prestressing internal resisting forces.

It is interesting to note that by considering a null increment of nodal movements ($a = 0$) in eqn (59)

$$\int P_{\mu} B_p^T ds = - \int \Theta^T p'' ds = - P_{\mu M} \Theta^T t_M + P_{\mu N} \Theta^T t_N \tag{62}$$

an alternative way to (47) arises for the evaluation of the work-equivalent node forces caused by prestress, as

$$P = \int_{r} P_{\mu} B_p^T ds. \tag{63}$$

### 3.9. Large displacements

As a particular case of the equations of kinematic compatibility,

$$\varepsilon_p = \varepsilon_p + \eta_p. \tag{59}$$

For $\eta_p$ component, eqn (50) can also be applied

$$\eta_p = C\eta. \tag{60}$$

Analogy with eqns (21 and 22) shows that the prestress contribution to the geometric stiffness follows from the term

$$\int_{r} \delta\eta_p \sigma_p ds. \tag{61}$$

As in Sec. 2.5, it is possible to construct a symmetric matrix $H$ to write (61) in the form

$$\delta a^T [A_p \int \sigma_{\mu 0} G^T H G ds ] a \tag{62}$$

so that the resulting expression for the prestressing geometric stiffness is given as

$$K_{op} = A_p \int \sigma_{\mu 0} G^T H G ds. \tag{63}$$

### 3.10. Strain determination and contribution to the stiffness for unbonded tendons

For unbonded tendons, the evaluation of the distribution of the axial deformation throughout the tendon requires kinematic considerations which involve the whole tendon length.

Neglecting the friction between the tendon and the duct will usually yield an acceptable approach for greased tendons. In such case, a uniform deformation will result from the total elongation of the tendon $u_p$

$$\varepsilon_p = \frac{u_p}{L}. \tag{64}$$

The total elongation $u_p$ may be obtained as the accumulated deformation of the surrounding concrete, $\varepsilon_p$

$$u_p = \int_{0}^{L} \varepsilon_p(s) d\sigma, \tag{65}$$

where $\varepsilon_p$ is obtained within each concrete element as

$$\varepsilon_p = C_B [x_p(s)] a. \tag{66}$$
Fig. 10. Multilinear stress-strain relationship for a prestressing tendon.

More sophisticated methods which consider the effect of the friction have been independently developed by Van Greunen [2], Hofstetter [7], Roca [8] and Murcia [16].

Although friction and anchorage confinement produce a certain sort of contribution from the unbonded tendon to the global stiffness, such a contribution will usually be significantly less important than for bonded tendons. Under certain peculiar situations, as for pure bending cases, the unbonded tendon contribution approximates to, or even reaches, that of the bonded tendon. Furthermore, the implementation of a procedure to form the unbonded tendon stiffness matrix, although conceivable, would not improve the realism of the model, since nonlinear problems do not require the usage of an exact global stiffness matrix. It follows that only computational advantages would result, but also the operational needs would be increased.

In the present formulation, a modified bonded tendon stiffness matrix is used as an approach

\[ \sigma_p = 1 - \log_{10} \left( \frac{\sigma_{pi}}{\sigma_p} - 0.55 \right) \]

for \( \frac{\sigma_p}{\sigma_p} \geq 0.55 \) (68)

where \( \sigma_p \) is the predicted final stress for an initial stress \( \sigma_{pi} \) and after \( t \) days. \( \sigma_{py} \) is the steel yielding stress.

This formula is only usable for constant imposed deformations. A further sophistication is needed to treat more complex situations, and, particularly, to deal with progressive or multiple loading processes. Figure 11 schematizes the full procedure as finally defined. Let \( f_{pi} \) be an initial stress. After a certain period \( t \), the stress will drop to \( f_R \), due to relaxation. For a modified stress \( f_P \), caused by an external instantaneous effect at time \( t_i \), the fictitious initial stress \( f_{PiI} \), associated to the curve that includes point \( (t, f_{Pi}) \) is obtained, and this last curve is used to model the later relaxation. This same process is recursively used for each stress increment. The total relaxation until time \( t \), is obtained as the sum of all stress decrements \( (f_{Rk}, k = 1, 2, \ldots, n) \).

3.13. Delayed prestressing force losses

Losses due to concrete creep and shrinkage are automatically evaluated through the partial constitutive equations introduced to deal with such time dependent phenomena.

Fig. 11. Relaxation losses evaluation for multiple loading.
Table 1. Summary of available expressions to include prestressing in a displacement formulation of the finite element method for beam, slab or shell analysis

<table>
<thead>
<tr>
<th>Concrete finite element</th>
<th>Prestressing segment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometric description</td>
<td>( x = x_0 + z \psi )</td>
</tr>
<tr>
<td>Displacements</td>
<td>( u = \Theta(x) a )</td>
</tr>
<tr>
<td>Strains</td>
<td>( \epsilon = Ba )</td>
</tr>
<tr>
<td>Node load</td>
<td>( R = \int \Theta^T r , dV )</td>
</tr>
<tr>
<td>Equivalent node ( y )</td>
<td>( P = \int \Theta^T p , ds )</td>
</tr>
<tr>
<td>Prestress force</td>
<td>( P = \int \Theta^T \bar{P} , ds )</td>
</tr>
<tr>
<td>Material</td>
<td>( K_0 = \int \mathbf{B}^T \mathbf{D} \mathbf{B} , dV )</td>
</tr>
<tr>
<td>Stiffness matrix</td>
<td>( K_{0p} = \int \mathbf{B}^T \mathbf{D}_{0p} \mathbf{B} , ds )</td>
</tr>
<tr>
<td>Geometric</td>
<td>( K_g = \int \mathbf{G}^T \mathbf{S} \mathbf{G} , dV )</td>
</tr>
<tr>
<td>Stiffness matrix</td>
<td>( K_{gp} = \int \sigma_g \mathbf{G}^T \mathbf{H} \mathbf{G} , ds )</td>
</tr>
<tr>
<td>Internal resisting</td>
<td>( \mathbf{R}' = \int \mathbf{B}^T \sigma , dV )</td>
</tr>
<tr>
<td>Nodal forces</td>
<td>( \mathbf{R}_p = \int \sigma \mathbf{B}_p^T , ds )</td>
</tr>
</tbody>
</table>

\[ \Delta \mathbf{R}_{\text{ps}} = - \int \mathbf{B}_p \Delta \sigma_p^T \, ds \]  \hspace{1cm} (69)

To compute the vector of internal resisting forces, the shell (or beam) stress vector \( \sigma \) must be substituted by a vector \( \sigma_p \).

\[ \sigma_p = \sigma_p \mathbf{C}^T. \]  \hspace{1cm} (71)

5. APPLICATION TO PARTICULAR STRUCTURAL CASES AND VERIFICATION

The implementation of the presented method to axisymmetric prestressed concrete shells, together with its verification by comparison with available experimental or numeric examples, has been successfully performed as described in an earlier paper [11].

The application to the nonlinear analysis of prestressed concrete shell structures of general geometry has also been achieved, showing the method to be both efficient and reliable. The steps to its particularization, as well as some aspects concerning the geometric definition, are discussed in another paper [10], with some examples used for the verification.

The application of this general method to a numerical model for the nonlinear analysis of space curved beams remains as a future perspective. Meanwhile, more numerical and experimental research is needed regarding the nonlinear behaviour of frame members under coupled sectional forces including torsion.

6. CONCLUSIONS

A formulation has been presented for the nonlinear material and geometric, instantaneous and long-term analysis of prestressed concrete structures. Such a
formulation is based on a discrete treatment of the prestressing tendons where both the prestress geometry and the mechanical effects are introduced in a consistent way to the displacement formulation of the finite element method. This consistency, together with some practical implementation considerations, make it possible to use the proposed formulation to include the numerical treatment of prestressing in many existing numerical models previously developed for reinforced concrete structures.

The general formulations have been applied to the specific analyses of axisymmetric shells, already presented, and to general geometry shells, which will be presented in a further paper. In both cases, the reliability and the efficiency of the formulation have been shown through the study of some available examples which were experimentally and analytically studied by other authors.

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REFERENCES